# STABILITY DOMAINS IN THE FEEDBACK GAINS SPACE FOR AN OVERHEAD CRANE WITH FLEXIBLE CABLE

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**Abstract:** The problem of stabilization of desired crane position and load at the flexible cable tip is considered. The stabilizing control is linear feedback. This control law depends on the platform position, velocity and integral of this position. It contains also the information about the angle between the vertical and the cable at the point of its connection to the platform and angular velocity, about the cable deformation. The time delay in the control loop is taken into account. Mathematical model of the system consists of differential equations with partial and ordinary derivatives.

The general equations describing boundaries of asymptotic stability domain are obtained. In the space of the feedback gains, the regions of asymptotic stability are designed for different particular cases. With feedback coefficients from these domains the desired equilibrium of the system is asymptotically stable.

**Keywords:** overhead crane, flexible cable, desired equilibrium, linear feedback control, boundary value problem, asymptotic stability, eigenvalues

## 1. Introduction

The problem of crane motion control is considered for example in monograph [1]. The problem of the crane stabilization is studied in the papers [2, 3] taking into account the flexibility of the cable. The stability problem is considered by using Lyapunov functions. This paper deals with the same problem of the crane stabilization. We examine the cable as the object with distributed parameters. Mathematical model of the system contains as in [2, 3] partial differential equation with boundary conditions. The principal goal of this paper is to find such linear feedback gains that the desired crane equilibrium would be asymptotically stable. Unlike the papers [2, 3], we study here the infinite spectrum of the boundary value problem and search such conditions that the real parts of all eigenvalues would be negative. The necessary and sufficient conditions are found in different particular cases more or less general. These results are formulated in the terminuses of the asymptotic stability domains in the space of the feedback gains.

The control problems for the systems with distributed parameters are examined in particular in monographs [4, 5], in the papers [6, 7].

The article is organized as follows: In Section 2 the motion equations of the system are described both with dimension and dimensionless variables; Section 3 contains the examined expression for linear feedback control and the stability problem statement, here the mathematical model of a crane with rigid cable is described as well; characteristic equation is found in Section 4, this section contains also the general equations for the boundaries of asymptotic stability domain; the stability domains in the feedback gains space for different cases are presented in Section 5, they are compared with stability domains for a crane with rigid cable; the influence of the cable deformation signal in the feedback is considered in Section 6; Section 7 contains our conclusion.

## 2. Mathematical Model

Let us consider the motion of an overhead crane along a horizontal straight line $N$ (Fig. 1).

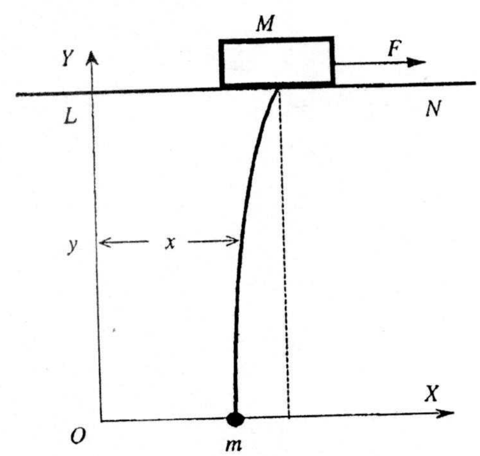


**Figure 1: Crane with flexible cable.**

This system consists of a motorized crane platform of mass $M$, flexible homogeneous cable of length $L$ with constant cross-section and a load of mass $m$. Let $X$ denotes the horizontal displacement of the platform from some fixed position, $F$ force applied to the platform from the motor, $P$ horizontal force applied to the platform from the cable, $\rho=const$ the linear cable density. We assume that the load is a material point (see Figure 1). Using these notations and assumptions we write the equations which govern a behavior of the described system:

$$M\ddot{X}(t) = F + P, \quad P = -(m+\rho L)g\frac{\partial x}{\partial y}(L,t), \tag{2.1}$$

$$\rho\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial y}\left[G(y)\frac{\partial x}{\partial y}\right], \quad G(y)=(m+\rho y)g$$

Here $x=x(y,t)$ is the horizontal displacement of the deformed cable point (dependent variable), $y$ its vertical coordinate (independent variable), $\frac{\partial x}{\partial y}$ the angle between the vertical and the tangent to the cable at the point $y$, $\frac{\partial x}{\partial y}(L,t)$ is the inclination of the cable tangent at the connection point to the platform with respect to the vertical, $G(y)$ the weight of the load together with "lower" part of the cable, $g$ is gravity acceleration. The origin of the inertial coordinate frame $XOY$ is placed on the horizontal crossed the load $m$ (Fig. 1). We neglect the vertical displacement of the load as a value of second order. The straight line $N$ is described in this coordinate frame $XOY$ by equation $y=L$. We assume here that the cable is non-stretching, its transversal and angular deformations are small. In the book [9], the motion equation of the hanging chain is written and it is similar to the third Eq. (2.1).

The following boundary conditions are added to the third Eq. (2.1):

$$x(L,t) = X(t), \quad \frac{\partial^2 x}{\partial t^2}(0,t) = g\frac{\partial x}{\partial y}(0,t) \tag{2.2}$$

The second condition (2.2) is the motion equation of the load along axis $OX$. In [3], instead this condition the equality $\frac{\partial x}{\partial y}(0,y) = 0$ is considered.

We introduce dimensionless variables $x^*$, $X^*$, $y^*$, $u$, $t^*$ using the formulas:

$$x = Lx^*, \quad X = LX^*, \quad y = Ly^*, \quad F = \frac{Mgm}{\rho L}u, \tag{2.3}$$

$$t = \tau t^* \quad (\tau^2 = \rho L^2/mg)$$

Substituting relations (2.3) into Eqs. (2.1), (2.2) and omitting the asterisks we rewrite them in the form

$$\ddot{x}(y,t) - [(\mu y + 1)x'(y,t)]' = 0, \tag{2.4}$$

$$\ddot{x}(0,t) - \mu x'(0,t) = 0, \quad \ddot{x}(1,t) = -\nu(1+\mu)x'(1,t) + u$$

$$(\mu = \rho L/m, \quad \nu = \rho L/M)$$

Here $'$ means the differentiation with respect to variable $y$.

## 3. The Problem Statement and Control

The boundary-value problem (2.4) has the solution

$$x(y,t) \equiv 0, \tag{3.1}$$

if $u=0$ (no control is applied). This solution describes the crane, the cable and the load in the equilibrium. Let this equilibrium be the desired (prescribed) one. Now let us find the control $u$ that ensures asymptotic stability of the desired equilibrium (3.1). We examine the stabilizing control as a linear feedback

$$T\dot{u} + u = -\beta_0 x(1,t) - \beta_1\dot{x}(1,t) - \beta_2\int_0^t x(1,\zeta)d\zeta -$$

$$-\delta_0 x'(1,t) - \delta_1\dot{x}'(1,t) - \sum_{n=1}^{N}\sigma_n x''(y_n,t) \tag{3.2}$$

Here $T\geq 0$ is dimensionless time delay in the control loop; constant values $\beta_0$, $\beta_1$, $\beta_2$ are the feedback gains for the platform position, its derivative and integral; the constants $\delta_0$ and $\delta_1$ are the feedback gains for the angle between the vertical and the cable at the point of its connection to the platform and angular velocity; $\sigma_n$ $(n=\overline{1,N})$ is the constant feedback gain with respect to the cable deformation at the point $y_n$. The first three terms in (3.2) describe the usual PID controller on the position of the platform.

The function (3.1) satisfies the Eqs. (2.4), (3.2). The linear boundary-value problem (2.4), (3.2) has an infinite number of eigenvalues (an infinite spectrum). The more concrete statement of the problem of asymptotic stability of the solution (3.1) is the following: it is required to find, in the space of the feedback (3.2) gains, the domain where all the eigenvalues $\lambda$ are such that $Re(\lambda)<0$.

### 3.1. *Rigid Cable*

Together with (2.4) we will examine the motion equations of the overhead crane with absolutely rigid cable. Using dimensionless variables (2.3) it is possible to write these equations [1] in the form

$$(1+\mu/2)\ddot{x} - (1+\mu/3)\ddot{\varphi} - (1+\mu/2)\mu\varphi = 0, \tag{3.3}$$

$$(\mu/\nu + \mu + 1)\ddot{x} - (1+\mu/2)\ddot{\varphi} = u\mu/\nu$$

Here $\varphi$ is the angle between the cable and the vertical calculated clockwise. Control law in this case has the form

$$T\dot{u} + u = -\beta_0 x - \beta_1\dot{x} - \beta_2\int_0^t x(1,\zeta)d\zeta - \delta_0\varphi - \delta_1\dot{\varphi} \tag{3.4}$$

because the cable deformation is neglected here.

We will compare lower the stability domains for the system (2.4), (3.2) and the system (3.3), (3.4).

## 4. Characteristic Equation

Let us search the solution of the system (2.4), (3.2) in the form

$$x(y,t) = Ce^{\lambda t}K(y),$$

where $C$ is a constant, $\lambda$ an eigenvalue, $K(y)$ an eigenfunction. For this eigenfunction we obtain the boundary-value problem

$$\lambda^2 K(y) - [(\mu y + 1)K'(y)]' = 0 \qquad (4.1)$$

$$\lambda^2 K(0) - \mu K'(0) = 0 \qquad (4.2)$$

$$[\lambda^2 K(1) + K'(1)(\mu + 1)v](T\lambda + 1)\lambda +$$
$$+(\beta_0\lambda + \beta_1\lambda^2 + \beta_2)K(1) + \lambda K'(1)(\delta_0 + \lambda\delta_1) +$$
$$+\lambda\sum_{n=1}^{N}\sigma_n K''(y_n) \qquad (4.3)$$

It is possible to reduce the Eq. (4.1) to Bessel equation using new independent variable instead $y$ [3, 8, 9]. But we will consider more simple case [10] when the mass of the cable is negligible with respect to the load mass

$$\mu << 1 \qquad (4.4)$$

Dimensionless variable $y$ changes in the interval $[0,1]$, therefore, under assumption (4.4) we neglect the term $\mu y$ in the Eq. (4.1) and it takes the form

$$K''(y) - \lambda^2 K(y) = 0 \qquad (4.5)$$

In the equality (4.3), we will use $v$ instead expression $(\mu+1)v$.

We represent the solution $K(y)$ of the system (4.5), (4.2), (4.3) in the form

$$K(y) = Ae^{\lambda y} + Be^{-\lambda y} \qquad (4.6)$$

where $A$ and $B$ are unknown constants. Substituting function (4.6) into boundary conditions (4.2), (4.3), we obtain a system of two linear homogeneous equations in constants $A$, $B$. The determinant of this system is characteristic one. It is always zero, if $\lambda=0$. But corresponding eigenfunction $K(y)$ is nonzero when $\beta_0 = \beta_2 = 0$ only. All nonzero eigenvalues $\lambda$ satisfy the following equation

$$\Delta(\lambda) = \lambda^2(T\lambda + 1)[\lambda R_1(\lambda) + vR_2(\lambda)] +$$
$$+(\beta_0\lambda + \beta_1\lambda^2 + \beta_2)R_1(\lambda) + \lambda^2(\delta_0 + \lambda\delta_1)R_2(\lambda) +$$
$$+\lambda^3\sum_{n=1}^{N}\sigma_n R_3(\lambda, y_n) = 0 \qquad (4.7)$$

$$(R_1(\lambda) = \lambda sh\lambda + \mu ch\lambda,\ R_2(\lambda) = \lambda ch\lambda + \mu sh\lambda,$$
$$R_3(\lambda, y) = \lambda sh(\lambda y) + \mu ch(\lambda y))$$

We see from the expression (4.7) that $\Delta(0) = \mu\beta_2$ and $\Delta(+\infty) = +\infty$. Therefore, if $\beta_2 < 0$, the Eq. (4.7) has a real root $\lambda > 0$ and the solution (3.1) is unstable. Let be $\Delta_1(\lambda) = \Delta(\lambda)/\lambda$. Then for $\beta_2 = 0$ we have $\Delta_1(0) = \mu\beta_0$ and $\Delta_1(+\infty) = +\infty$. Therefore, if $\beta_2 = 0$ and $\beta_0 < 0$ (positive position feedback), the Eq. (4.7) has a root $\lambda > 0$ and solution (3.1) is unstable. Let be now $\Delta_2(\lambda) = \Delta_1(\lambda)/\lambda$. Then for $\beta_2 = \beta_0 = 0$ we have $\Delta_2(0) = \mu\beta_1$ and $\Delta_2(+\infty) = +\infty$. Therefore, if $\beta_2 = \beta_0 = 0$ and $\beta_1 < 0$ (negative damping), the Eq. (4.7) has a root $\lambda > 0$ and solution (3.1) is not stable. These conclusions we use below when study the asymptotic stability domains in the feedback gains space.

The necessary and sufficient conditions of asymptotic stability are obtained below by method of D-subdivisions of Neimark [11]. To this end, we substitute $\lambda = i\omega$, where $i$ is imaginary unit and $\omega$ real value, into the Eq. (4.7) and equate both the real and imaginary parts to zero

$$T\omega^3[\omega S_1(\omega) + vS_2(\omega)] + (\beta_2 - \beta_1\omega^2)S_1(\omega) +$$
$$+\omega^3\delta_1 S_2(\omega) = 0$$
$$(4.8)$$
$$\omega^2[\omega S_1(\omega) + vS_2(\omega)] - \beta_0\omega S_1(\omega) + \omega^2\delta_0 S_2(\omega) +$$
$$+\omega^3\sum_{n=1}^{N}\sigma_n S_3(\omega, y_n) = 0$$

$$(S_1(\omega) = R_1(i\omega) = \mu\cos\omega - \omega\sin\omega,$$
$$S_2(\omega) = -iR_2(i\omega) = \omega\cos\omega + \mu\sin\omega,$$
$$S_3(\omega, y) = R_3(i\omega, y) = \mu\cos(\omega y) - \omega\sin(\omega y))$$

Eqs. (4.8) define, in parameter space of the system, the image of the imaginary axis $\lambda = i\omega$, $-\infty < \omega < \infty$. The stability domain boundary consists of parts of surface (4.8). Relations (4.8) remain unchanged, if $\omega$ is replaced by $-\omega$. Therefore, the boundaries of an asymptotic stability domain can be obtained using the Eqs. (4.8) with $0 \le \omega < \infty$ only.

## 5. Stability Domains

We will construct the regions of asymptotic stability analytically for different particular cases.

Let us consider initially the usual PD controller. It means that we suppose first that

$$T=0,\ \beta_2 = 0,\ \delta_0 = 0,\ \delta_1 = 0,\ \sigma_n = 0\ \ (n = \overline{1,N}) \qquad (5.1)$$

If $\beta_2 = 0$, i.e. the integral in the control law (3.2) is absent, we must first of all divide both sides of the characteristic Eq. (4.7) by $\lambda$ and of Eqs. (4.8) by $\omega$. Instead of (4.8) under conditions (5.1) we obtain simple equations

$$\beta_1\omega S_1(\omega) = 0,\ (\beta_0 - \omega^2)S_1(\omega) - v\omega S_2(\omega) = 0 \qquad (5.2)$$

Setting in (5.2) $\omega=0$, we obtain equation $\beta_0 = 0$. Each function $S_1(\omega)$ and $S_2(\omega)$ has infinite number of zeros for any $\mu$. The zeros of these functions are different and moreover alternates. Thus, with $\omega>0$ the Eqs. (5.2) are valid if $\beta_1 = 0$ only, and these equations can be rewritten in the form

$$\beta_0 = \omega^2 + v\omega S_2(\omega)/S_1(\omega),\ \beta_1 = 0 \qquad (5.3)$$

As $\omega$ changes from $0$ to $+\infty$ the point (5.3) traverses the axis $\beta_1 = 0$ from $-\infty$ to $+\infty$ an infinite number of times. Hence the boundaries of asymptotic stability domain, if there is one, belong to the straight lines $\beta_0 = 0$ and $\beta_1 = 0$. It follows from the Section 4 that outside the region $\beta_0 \ge 0$, $\beta_1 \ge 0$, the Eq. (4.7) has a root $\lambda$ with $Re\lambda>0$. Thus, under conditions (5.1) the asymptotic stability region, if it exists, can be described by inequalities

$$\beta_0 > 0, \ \beta_1 > 0 \qquad (5.4)$$

Let us prove by method described in [15-17] that the stability domain $D$ is actually an open region (5.4). We multiply both sides of the Eq. (4.5) by the conjugate function $\overline{K(y)}$ and integrate from zero to one. Using boundary conditions (4.2), (4.3), we obtain

$$\lambda^2\left[v\int_0^1 \overline{K(y)}K(y)dy + \overline{K(1)}K(1) + \mu v\,\overline{K(0)}K(0)\right] +$$

$$\lambda\beta_1 \overline{K(1)}K(1) + \beta_0 \overline{K(1)}K(1) + v\int_0^1 \overline{K'(y)}K'(y)dy = 0$$

In the domain (5.4) the coefficients of this quadratic equation in $\lambda$ are non-negative. Thus, in this domain all eigenvalues have non-positive real parts. Since $Re\lambda=0$ only if $\beta_0 = 0$ or $\beta_1 = 0$, inside that region for all $\lambda$, $Re\lambda<0$. The assertion is proved.

In the case (5.1), also the region of asymptotic stability of the equilibrium state of system (3.3), (3.4) can be described by inequalities (5.4). It can be proved using Hurwitz conditions with assumption (4.4) and without it. Thus the flexibility of the cable has no influence on the stability domain in the case (5.1).

Suppose now that $T>0$ and consider more general case than (5.1), namely

$$\beta_2 = 0, \ \delta_0 = 0, \ \delta_1 = 0, \ \sigma_n = 0 \ \ (n=\overline{1,N}) \qquad (5.5)$$

If $\beta_2 = 0$, we divide firstly the Eqs. (4.8) by $\omega$. Then substituting into the Eqs. (4.8) first $\omega=0$, and then $\omega>0$, we find that the boundary of the stability domain is made up of segments of the straight line $\beta_0 = 0$ and the straight line

$$\beta_0 = \omega^2 + v\omega S_2(\omega)/S_1(\omega),$$
$$\beta_1 = T\omega^2 + Tv\omega S_2(\omega)/S_1(\omega) \quad (0\le\omega<\infty) \qquad (5.6)$$

The parametric Eqs. (5.6) ($\omega$ is the parameter) describe a straight line, since they imply that

$$\beta_1 = T\beta_0 \qquad (5.7)$$

As the quantity $\omega$ changes from $0$ to $\infty$, the point (5.6) traverses the straight line (5.7) an infinite number of times. Let $D(T)$ (see Figure 2) denotes the open domain

$$\beta_0 > 0, \ \ \beta_1 > T\beta_0 \qquad (5.8)$$

In the domain $D(T)$, all the eigenvalues are such that $Re\lambda\neq0$, since $Re\lambda=0$ only if $\beta_0 = 0$ or $\beta_1 = T\beta_0$. Let us consider the set $D(T)$ in the space of the three parameters $\beta_0$, $\beta_1$, $T$ for $0\le T<\infty$. As $T \to 0$ we have $D(T)\to D$, where $D$ is the asymptotic stability domain in the case (5.1). It follows from Rouche theorem [14] that the eigenvalues $\lambda$ are continuous functions of $T$. Consequently, the real parts of all eigenvalues are negative, if and only if $\beta_0, \beta_1 \in D(T)$.

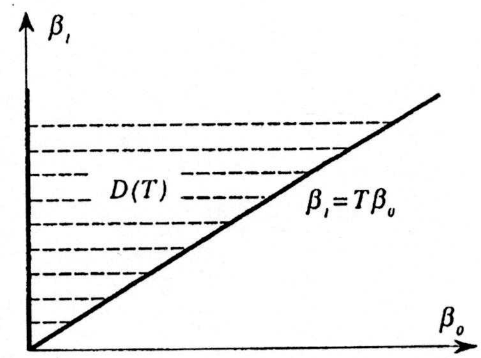


**Figure 2: Stability domain in the case (5.5)**

In the case (5.5), the stability domain of the equilibrium of the system (3.3), (3.4) is described by inequalities (5.8) as well (under condition (4.4) and without it). Hence, as in the case (5.1), the elasticity of the cable has no influence on the stability.

Let us study now more general case than (5.5). Suppose that $T>0$ and $\beta_2 \neq 0$, but as before

$$\delta_0 = 0, \ \delta_1 = 0, \ \sigma_n = 0 \ \ (n=\overline{1,N}) \qquad (5.9)$$

If $\beta_2 < 0$, solution (3.1) of system (2.4), (3.2) is unstable (see Section 4). We shall assume therefore that $\beta_2 > 0$. Let us choose some positive quantity $\beta_2$ and construct the stability domain $D(T, \beta_2)$ in the plane of the coefficients $\beta_0$, $\beta_1$.

The boundary of the region $D(T, \beta_2)$ is described by parametric equation similar to (5.6) but with additional term $\beta_2/\omega^2$ in the second of them

$$\beta_0 = \omega^2 + v\omega S_2(\omega)/S_1(\omega),$$
$$\beta_1 = T\omega^2 + Tv\omega S_2(\omega)/S_1(\omega) + \beta_2/\omega^2 \qquad (5.10)$$
$$(0 \le \omega < \omega_1)$$

Here $\omega_1$ is the first positive root of the function $S_1(\omega)$ $(\omega_1 < \sqrt{\mu})$. As $\omega \to 0$, the curve (5.10) tends asymptotically to the axis $\beta_1$, and as $\omega \to \omega_1$, tends asymptotically to the straight line

$$\beta_1 = T\beta_0 + \beta_2/\omega_1^2 \qquad (5.11)$$

This line is parallel to the line (5.7) and shifted upward relative to it by the value $\beta_2/\omega^2$. It is obviously that the domain $D(T, \beta_2)$ is located inside the domain $D(T)$. If the time delay $T$ in the control loop or (and) coefficient $\beta_2$ increases, the region $D(T, \beta_2)$ decreases. The asymptotic stability domain $D(T, \beta_2)$ is shown in Figure 3. Its boundary is located between two asymptotes – axis $\beta_1$ and line (5.11), and similar to a hyperbole.

In the case (5.9) the degree of the characteristic polynomial of system (3.3), (3.4) equals six. The Hurwitz inequalities for this polynomial are cumbersome, therefore we do not write them here. But we can show that the functions $\beta_0(\omega)$, $\beta_1(\omega)$ from (5.10) satisfy these inequalities strictly and domain $D(T, \beta_2)$ is located strictly inside the stability domain of the system with rigid cable. Hence, unlike the previous cases, in the case (5.9) the asymptotic stability domain decreases, if we take into account the cable flexibility. Thus, the study of the crane stability neglecting the cable elasticity can be misleading.

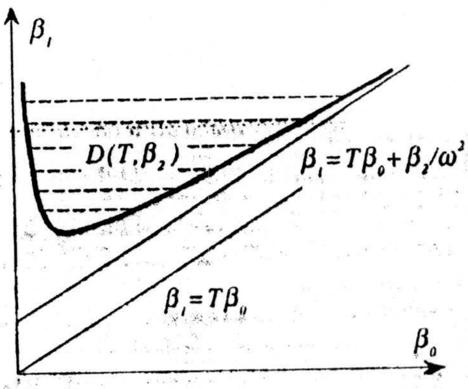


**Figure 3: Stability domain in the case (5.9)**

Suppose now that $\delta_0 \neq 0$ and take more general case than (5.1), namely,

$$T=0,\ \beta_2=0,\ \delta_1=0,\ \sigma_n=0\ (n=\overline{1,N}) \quad (5.12)$$

In this case, instead of (4.8) we obtain the equations

$$\beta_1\omega S_1(\omega)=0$$
$$(\beta_0-\omega^2)S_1(\omega)-(\nu+\delta_0)\omega S_2(\omega)=0 \quad (5.13)$$

Under assumption $\omega=0$ we obtain equation $\beta_0=0$. The equalities (5.13) exist for all $\omega>0$, if and only if $\beta_1=0$. Let $\omega=\omega_k$, where $\omega_k$ (k=1, 2, ...) are positive zeros of the function $S_1(\omega)$, then equalities (5.13) exist, if

$$\nu+\delta_0=0 \quad (5.14)$$

Prove that the asymptotic stability domain in the space of the gains $\beta_0$, $\beta_1$, $\delta_0$ is the set

$$\beta_0>0,\ \beta_1>0,\ \delta_0>-\nu \quad (5.15)$$

Under conditions (5.1), in the domain (5.4) which is located inside the domain (5.15), the real parts of all eigenvalues are negative. At the same time in the domain (5.15), there is no eigenvalues with zero real parts. It follows from Rouche theorem [14] that the eigenvalues $\lambda$ are continuous functions of $\delta_0$. Therefore, in the domain (5.15) the eigenvalues with positive or zero real parts are absent. Under the conditions (5.12) the characteristic Eq. (4.7) takes the form

$$\Delta(\lambda)=(\lambda^2+\beta_1\lambda+\beta_0)R_1(\lambda)+\lambda(\nu+\delta_0)R_2(\lambda)=0 \quad (5.16)$$

Show now that the Eq. (5.16) gets eigenvalues with positive real parts, when the feedback gains forsake domain (5.15) crossing the boundary (5.14). Under condition (5.14) ($\delta_0=-\nu$) this equation has the roots $\lambda_k=i\omega_k$ $(k=1,\ 2,\ \ldots)$ and two roots of equation $\lambda^2+\beta_1\lambda+\beta_0=0$. Let be $\delta_0=-\nu-\Delta\delta_0$ and $\lambda=\lambda_k+\Delta\lambda_k$; here $\Delta\delta_0>0$ and $\Delta\lambda_k$ are small variations. Linearise the Eq. (5.16) in a vicinity of the point $\delta_0=-\nu$, $\lambda=\lambda_k$

$$(\lambda_k^2+\beta_1\lambda_k+\beta_0)R_1'(\lambda_k)\Delta\lambda_k=\lambda_k\Delta\delta_0 R_2(\lambda_k)$$

From this linear equation we obtain

$$Re\Delta\lambda_k=\frac{\beta_1\omega_k^2(\mu^2+\omega_k^2)}{[\beta_1^2\omega_k^2+(\beta_0-\omega_k^2)^2](\omega_k^2+\mu^2+\mu)}\Delta\delta_0$$

If $\Delta\delta_0>0$, then $Re\Delta\lambda_k>0$. Hence, the infinite number of eigenvalues becomes with positive real part, when the point in the feedback gains space crosses the boundary (5.14). The eigenvalues $\lambda$ are continuous functions of the system parameters and they can be with zero real part on the boundaries of the domain (5.15) only. Therefore outside the region (5.15), there are eigenvalues with positive real parts and (5.15) is the asymptotic stability domain.

Using Hurwitz criteria we can show that for the system (3.3), (3.4) the stability domain is described by inequalities (5.15) as well.

We can consider more general case than (5.12), namely, when $\delta_1 \neq 0$, but

$$T=0,\ \beta_2=0,\ \sigma_n=0\ (n=\overline{1,N}) \quad (5.17)$$

In this case, the asymptotic stability domain located strictly inside the stability domain for the crane with rigid cable. Hence, it is better to take into account the cable flexibility in the case (5.17).

## 6. Influence of the Cable Deformation Feedback

Let us consider the case when control law (3.2) contains the information about the cable deformation.

Suppose that

$$T>0,\ \beta_2=0,\ \delta_0=0,\ \delta_1=0,\ \sigma_1\neq 0,\ y_1=1,$$
$$\sigma_n=0\ (n=\overline{2,N}) \quad (6.1)$$

These relations mean that the signal about the cable deformation in the point of its connection to the platform is taken into account.

In the case (6.1), the stability domain $D(T,\sigma_1)$ is located in the first quadrant of the plane $\beta_0$, $\beta_1$. It is bounded by the semi-axis $\beta_0=0$, $\beta_1\geq 0$ and the curve

$$\beta_0=\omega^2(1+\sigma_1)+\nu\omega S_2(\omega)/S_1(\omega)$$
$$\beta_1=T\omega^2+T\nu\omega S_2(\omega)/S_1(\omega)\quad (0\leq\omega<\omega_1) \quad (6.2)$$

If $\sigma_l = 0$, the curve (6.2) becomes the straight line (5.6) or (5.7). If $\sigma_l > 0$, the curve (6.2) is located lower than the line (5.7) and $D(T) \subset D(T,\sigma_l)$; the domain $D(T,\sigma_l)$ is greater than domain $D(T)$ (5.8) (see Figure 4).

If the gain $\sigma_l$ increases, the curve (6.2), remaining in the first quadrant, lowers and the domain $D(T,\sigma_l)$ increases. As $\sigma_l \to \infty$, the curve (6.2) tends to the semi-axis $\beta_l = 0$, $\beta_0 \geq 0$ and, in spite of the time delay $T$, the domain $D(T,\sigma_l)$ tends to the stability domain $D$ (5.4) for the cases of flexible or rigid cable under conditions (5.1). Note, that it is for ideal deformation signal.

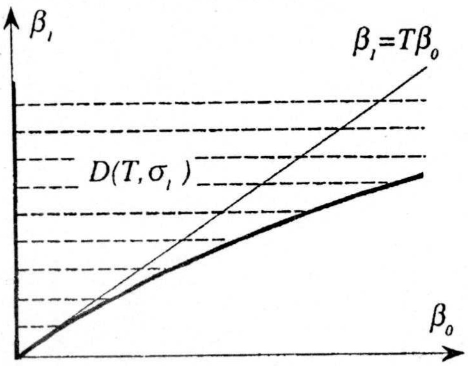


Figure 4: Stability domain in the case (6.1)

Thus, we can enlarge the asymptotic stability domain by introducing the deformation signal into the feedback control. So, the information about the cable deformation in some sense compensates the time delay in the control loop.

## 7. Conclusion

The main goal of this paper is to describe the complete set of feedback gains to ensure the asymptotic stability of the equilibrium of the overhead crane with a load at the flexible cable tip. We have obtained the general analytical relations describing the boundaries of the stability domain. Using obtained analytical formulas we have found the structure of stability domain in different cases. The designed stability domains are compared with corresponding domains for the crane with rigid cable. In all considered cases the stability domain for the crane with flexible cable belongs to, and in some cases less than, the domain for the crane with rigid cable. Hence, the cable elasticity should not be ignored in the problem of stability. The stability domain can be prolonged by using the deformation signal in the control law.

Using our formulas it is possible as well to construct numerically in the space of feedback coefficients the boundary of stability domain for any given parameters of crane, cable and load.

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