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SOME EXPERIMENTAL FORCE ANALYSIS FOR AUTOMATION OF EXCAVATION BY A BACKHOE

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Abstract Automating the excavation function of an excavator is a complex control problem, because of the nature of the interaction between the cutting tool and the medium and the many parameters that are involved in the process. A fundamental question that is not yet resolved is the choice of parameters to be used in a feedback loop for the control of the process, and the control algorithm. Based on a previous proposition for the control algorithm to be used for this purpose, knowledge about the forces involved may be used for adjusting the motion of the bucket. In this paper, some results of experimental measurements of the excavation forces are presented. These forces cannot be measured directly on the bucket, but the kinematic and dynamic relations can serve for this purpose. The paper talks about the development of these relations, and then the results are presented and discussed.

Keywords: Automation, excavation, force measurement, kinematics, backhoe

INTRODUCTION

Automation of excavation necessarily implies that an excavator carries out its function without being operated by a person. This function consists of one or more of cutting, digging, scooping and penetration into the medium to be excavated. Excavating machines are of different form, depending on the environment and/or the application. For example a backhoe is most suitable for excavating below ground, whereas a power shovel is designed to load its bucket from a higher level. With all excavating machines, the bucket or the excavating tool must interfere with the material to be excavated and, necessarily, must overcome the inherent resistance that it experiences from the medium. In other words, the tool must overcome the resistance from the environment it is dealing with. This resisting force, however, is a function of a large number of parameters dependent upon the medium, the tool and the environment. It also depends on the motion of the excavator's tool. In this sense, the value and the direction of the resistive force has a significant variation during the course of an excavation task, since it has a stochastic nature and, as yet, has not been mathematically modeled.

Considering the fact that the normally curvilinear motion of the bucket is provided by a set of actuators, it is essential that the required force by each actuator be furnished at each instant. This implies that each actuator must provide sufficient power in order to maintain the motion of the bucket. In practice, this is possible only if an enormous amount of power is available at each actuator, which is generally uneconomic. In the case of a human operator, a skilled person continuously adjusts the tool motion, by using his intelligence and experience, so that the final goal of excavation is accomplished. In fact, he adjusts the motion when he feels that the tool motion is not what was intended. Otherwise, not only the task is not carried out properly, but the machine elements may get damaged, too.

Various approaches have been suggested on the matter and numerous work has been reported. Some of the reported works are listed in the References of the present paper. The main issue in automated excavation is successfully filling the bucket. There exist two extreme cases, both of which are unsatisfactory. Either the bucket gets stuck in a situation where an increase in force is useless, or the bucket continues its motion without being loaded. In the first case any more effort to move the bucket has no effect other than damaging the system (slipping and eroding the tires in a front-end loader, for instance). The second case, obviously, is not desirable, either.

In previous work [3] it was suggested that, by finding the envelope of the maximum and minimum excavation force during a specific excavation task, one can avoid the two extreme cases. By "specific" it is meant that a given bucket (fixed physical dimensions) has a well defined motion (fixed trajectory) to excavate from a medium that by a common judgement is fixed (water content, density, size and other characteristics have no dramatic variations), and the environmental conditions (temperature and gravity) do not introduce phase changes (freezing, for instance). In this way, by measurement of the excavation force as a feedback parameter, the bucket motion can be readjusted anytime the experienced force is outside of its expected limits.

In this paper, some results of experimental measurement of excavation forces are presented. For this purpose, first, the kinematic and force relationships for a typical backhoe excavator are derived.

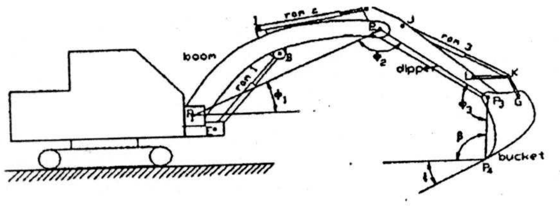

Figure 1

KINEMATIC MODELING

Figure 1 shows the schematic of the system under study and certain variables and definitions. System kinematic relationships are necessary for the transformation of the desired cutting path and velocities in individual joint movements. The Denavit-Hartenberg coordinates system assignment will be employed. For reasons of simplicity we will ignore the slewing of the cab about the vehicle tracks. The other joint motions take place, more or less, in plane. Thus a planar analysis is sufficient, and the cutting edge will be represented by a point. The choice of coordinate systems is similar to that of the work of Vähä et al [12], except for the deletion of the first coordinate system $X_0Y_0Z_0$ (for the inclusion of slew motion) which in our case is the same as the shown $X_1Y_1Z_1$. Thus the transformation between the reference frame and the first frame is the identity matrix.

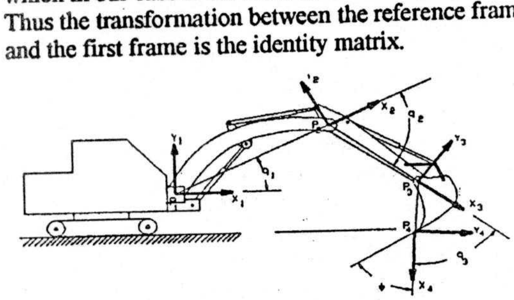

Figure 2

Figure 2 shows the various coordinate systems. The associated transformation matrices are as follows:

$$A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

$$A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

$$A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

where S and C stand for sin (.) and cos (.), respectively, and the subscripts 1-3 denote the assigned number of a joint and its associated coordinate system. The parameter a's are the joint lengths based on the D-H parameter definition; the twist angles are all zero for the simple planar robot arm of the excavator. The joint variables are represented by q and the other angles, required for geometric or mathematical relationships, are as shown in the figures 3 and 4.

For consistency with previous work, some additional variables are defined such as the manufacture's maximum and minimum values of each joint angle. The lengths l_1 to l_{25} are geometric dimensions. From these, only those pertinent to the studies and used in this paper are shown. .Based on these definitions, the corresponding values for the joint lengths are as follow:

$$a_1 = l_1, \quad a_2 = l_2, \quad and \quad a_3 = l_3 \quad (4)$$

The coordinates of the cutting edge of the bucket with respect to the reference coordinate system are given by

$$x = a_1C_1 + a_2C_{12} + a_3C_{123} \quad (5)$$

and

$$y = a_1S_1 + a_2S_{12} + a_3S_{123} \quad (6)$$

and the orientation of the bucket,

$$\beta = q_1 + q_2 + q_3 \quad (7)$$

where $C_1 = \cos(q_1)$, $C_{12} = \cos(q_1 + q_2)$ and $C_{123} = \cos(q_1 + q_2 + q_3)$, and likewise for S_1, S_{12} and S_{123}. However, if the angle of the bucket teeth, φ is required, the constant value of the angle ψ must be added to it. This orientation is with respect to the untilted body of the backhoe. If the machine has an angle μ with the horizontal plane and the angle of the bucket with a horizontal plane is desired, then the angle μ must be added to the said values. The positive direction of all the angles are counterclockwise for the configuration of the excavator shown in the figures.

The Jacobian matrix is used to relate the velocities and the forces at a given point of interest on a robot arm with respect to a desired coordinate frame. The Jacobian matrix for the system under study, defining the motion of the tool point(the cutting edge of the bucket),with respect to frame 1 is as follows:

$$J = J_4^1 = \begin{bmatrix} -a_3\sin(q_1+q_2+q_3) - a_2\sin(q_1+q_2) - a_1\sin q_1 & -a_3\sin(q_1+q_2+q_3) - a_2\sin(q_1+q_2) & -a_3\sin(q_1+q_2+q_3) \\ a_3\cos(q_1+q_2+q_3) + a_2\cos(q_1+q_2) + a_1\cos q_1 & a_3\cos(q_1+q_2+q_3) + a_2\cos(q_1+q_2) & a_3\cos(q_1+q_2+q_3) \\ -1 & -1 & -1 \end{bmatrix} \quad (8)$$

INVERSE KINEMATICS

Inverse kinematics implies finding the values for the joint angles, q_1, q_2 and q_3 when the position and orientation of the cutting edge is known with respect to the reference frame $X_0Y_0Z_0$. It follows from equations (5) to (7) that when β is known, then q_{123} is known and having x and y leads to:

$$a_1 \cos q_1 + a_2 \cos(q_1 + q_2) = x \qquad (9)$$

and

$$a_1 \sin q_1 + a_2 \sin(q_1 + q_2) = y \qquad (10)$$

These are two equations in two unknowns; their solution can be easily found, as shown in Appendix B. Once solved, then the values of q_1, q_2 and q_3 are readily determined.

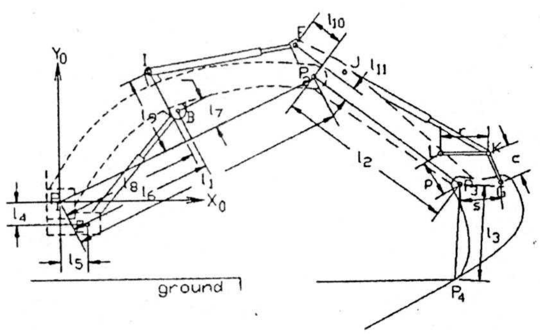

Figure 3

The other necessary calculations are the relationships between the ram lengths (actuator variables) and the position and orientation of the cutting teeth of the bucket, and the ram forces and the corresponding cutting/loading forces the bucket must exert to its environment.

The relationship between the length of the first actuator and its corresponding joint variable q_1 is

$$ram\ 1 = \sqrt{l_b^2 + l_4^2 + l_5^2 - 2l_b[l_5 \cos(q_1 + \eta) + l_4 \sin(q_1 + \eta)]} \qquad (11)$$

where η is the angle BP_1P_2, which is constant, and

$$l_b = \sqrt{l_6^2 + l_7^2}$$

Equation (11) can be used, also, to find the magnitude of q_1 in terms of the actuator length. The corresponding equation is in the general form of

$$m_0 \sin(q_1 + \eta) + n_0 \cos(q_1 + \eta) = h_0 \qquad (12)$$

where

$$m_0 = \frac{l_4}{2\sqrt{l_6^2 + l_7^2}}; \quad n_0 = \frac{l_5}{2\sqrt{l_6^2 + l_7^2}} \quad and \quad h_0 = l_4^2 + l_5^2 + l_6^2 + l_7^2 - (ram1)^2$$

This equation has generally two answers, but only one of them is acceptable for the system under study. These answers are

$$q_1 = \tan^{-1}(h_0, \pm\sqrt{m_0^2 + n_0^2 - h_0^2}) - \tan^{-1}(n_0, m_0) - \eta \qquad (13)$$

In equation (13), $\tan^{-1}$ stands for atan2 function.

In a similar manner, the length of the actuator 2 in terms of angle q_2 is

$$ram2 = \sqrt{(l_1 - l_8)^2 + l_9^2 + l_f^2 - 2l_f[(l_1 - l_8)\cos(\gamma - q_2) + l_9 \sin(\gamma - q_2)]}$$

where γ is the constant angle FP_2P_3 and l_f is the distance between points P_2 and F, and in terms of the dimensions shown in figure 3 is

$$l_f = \sqrt{l_{10}^2 + l_{11}^2} \qquad (15)$$

The magnitude of angle q_2 in terms of the ram length is found from

$$q_2 = \gamma - \tan^{-1}(d, \pm\sqrt{l_9^2 + (l_1 - l_8)^2 - d^2}) - \tan^{-1}[(l_1 - l_8)/l_9] \qquad (16)$$

where

$$d = \frac{(l_1 - l_8)^2 + l_9^2 + l_f^2 - (ram\ 2)^2}{2l_f}$$

The relationships for the third joint are more complicated, because of the four-bar linkage. Changing the length of ram 3 leads to changes in the angles of the linkage, which govern the bucket motion. The four-bar linkage is shown in figure 4. The lengths of the sides are represented by c, p, r and s. In a classical linkage the input angle is λ and the output angle is α. In our case, however, we are interested in the relationship between the actuator length and its joint angle q_3. The length of ram 3 can be defined as

$$ram3 = \sqrt{l_{24}^2 - 2l_{24}[(l_{22} - l_{27})\sin\lambda_3 - (l_{21} - l_{28})\cos\lambda_3] + (l_{22} - l_{27})^2 + (l_{21} - l_{28})^2} \qquad (17)$$

where all the elements are as shown in figure 4. Here, instead of λ another angle λ_3 (see figure 4) is employed. When necessary, the following relationships can be used.

$$\lambda_1 = \tan^{-1}\frac{l_{21}}{l_{22}} \quad and \quad \lambda_2 = \pi/2 - \lambda \qquad (18)$$

$$\text{and} \quad \lambda_3 = \pi - \lambda_2 - \lambda \qquad (19)$$

$$\text{so that } \sin\lambda_3 = \cos(\lambda_1 - \lambda) \quad and \quad \cos\lambda_3 = \sin(\lambda - \lambda_1) \qquad (20)$$

The following relationship holds between λ_3 and α :

$$(l_{22} - r\sin\lambda_3)^2 + (l_{21} + rcs\lambda_3)^2 + 2s\sin\alpha(l_{22} - r\sin\lambda_3)$$
$$- 2s\cos\alpha(l_{21} + rcs\lambda_3) = c^2 - s^2 \qquad (21)$$

From this equation we can derive α in terms of λ_3 or vice versa. In order to find α the resulting equation is in the form of

$$m_1 \sin\alpha + n_1 \cos\alpha = h_1 \qquad (22)$$

where $m_1 = -(l_{21} + r\cos\lambda_3)$ $\quad n_1 = (l_{22} - r\sin\lambda_3)$

and

$$h_1 = \frac{1}{2s}(c^2 - s^2 - r^2 - p^2) - \frac{r}{s}(l_{21}\cos\lambda_3 - l_{22}\sin\lambda_3)$$

The solution to this equation is given in appendix A. The same solution applies for the case of determining λ_3 in terms of α, whose equation is

$$m_2 \sin\lambda_3 + n_2 \cos\lambda_3 = h_2 \qquad (23)$$

where

$$m_2 = -(l_{22} + s\cos\alpha) \qquad n_2 = (l_{21} - s\sin\alpha)$$

and

$$h_2 = \frac{1}{2r}\left(c^2 - s^2 - r^2 - p^2\right) - \frac{s}{r}\left(l_{21}\cos\alpha - l_{22}\sin\alpha\right)$$

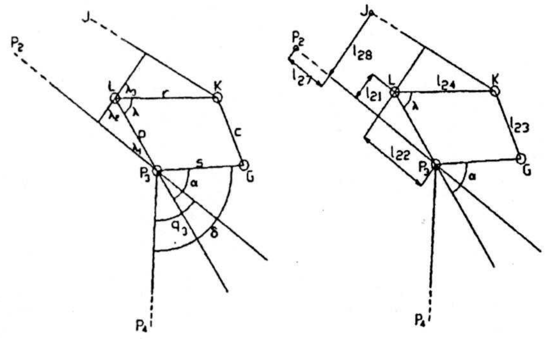

Figure 4

Also, as can be seen from figure 4 angle δ is constant for a given bucket, and the variation of α and the joint variable q_3 are equal and in the same direction. That is to say, α and q_3 have the same rate of change and their magnitude is different by a constant. That is

$$\alpha = \delta + \lambda_1 + q_3 \qquad (24)$$

This is more evident when the value of q_3 is zero, so that the point P_4 is in line with P_2P_3. It also follows from equation (17) that the value of angle λ_3 in terms of the ram length is determined from

$$(l_{22} - l_{27})\sin\lambda_3 - (l_{21} - l_{28})\cos\lambda_3 = \frac{1}{2l_{24}}\left[l_{24}^2 + (l_{22} - l_{27})^2 + (l_{21} - l_{28})^2 - ram3^2\right] \qquad (25)$$

This equation is again in the form of equations (22) and (23), and has a pair of solutions. From the above equations, the length of the actuator, *ram3*, can be found in terms of q_3 and vice-versa.

FORCE/TORQUE RELATIONSHIPS

For our studies, the Jacobian matrix is most suitable for the force/torque relationships between the reference coordinates system and the joint efforts, that is, the force torque in the joints. This relationship is

$$\tau = J^T F \qquad (26)$$

where F is vector of the forces requirement in the Cartesian coordinates, consisting of the force in the X_0-direction, Y_0-direction and the moment about Z_0 axis, and τ is the associated joint efforts (force or torque). It is necessary, furthermore, to find the relationships between the joint torques and the forces in the actuators, the three hydraulic cylinders. If the three forces in the actuators are denoted by f_1, f_2 and f_3, then we have the following relationships:

$$\tau_1 = \frac{l_b\left[l_5\sin(q_1+\eta) - l_4\cos(q_1+\eta)\right]}{\sqrt{l_4^2 + l_5^2 + l_b^2 - 2l_b\left[l_4\sin(q_1+\eta) + l_5\cos(q_1+\eta)\right]}} f_1 \qquad (27)$$

$$\tau_2 = \frac{l_r\left[l_9\cos(\gamma - q_2) - (l_1 - l_8)\sin(\gamma - q_2)\right]}{\sqrt{(l_1 - l_8)^2 + l_9^2 + l_r^2 - 2l_r\left[(l_1 - l_8)\cos(\gamma - q_2) + l_9\sin(\gamma - q_2)\right]}} f_2 \qquad (28)$$

$$\tau_3 = \frac{l_{28}\left[l_{24}\sin\lambda_3 - (l_{22} - l_{27})\right] + l_{27}\left[l_{24}\cos\lambda_3 + (l_{21} - l_{28})\right]}{\sqrt{l_{24}^2 - 2l_{24}\left[(l_{22} - l_{27})\sin\lambda_3 - (l_{21} - l_{28})\cos\lambda_3\right] + (l_{22} - (l_{22} - l_{27}))^2 + (l_{21} - l_{28})^2}} f_3 \qquad (29)$$

From the above equations, one can determine the joint torques in terms of the ram forces, Then from equation (26) the forces at the cutting edge of the bucket (tool point) are determined. The inverse of this process determines the required ram forces to provide the resistive forces on the bucket.

EXPERIMENTAL RESULTS

The following experiments were carried out at Lancaster University using LUCIE the Lancaster University Computerized Intelligent Excavator. The experimental work consists of the measurement of the ram forces during a number of excavation tasks on the same soil. In this respect, all the parameters related to the machine and the medium remain constant. Also, the environment factors such as the temperature and the slope of the terrain are almost the same. As a result, the measured values reflect a true representation of the variation of the excavation forces. As expected, this force has a stochastic nature and varies considerably during the process of excavation.

From the ram forces, the excavation force at the edge of the bucket can be calculated by using equations (27)-(29), first, finding the corresponding joint forces; then, using equation (26), relating the joint forces to the three components of the excavation force. In our experiments, only the ram forces for the boom and the dipper are available, because only two pressure sensors were installed at the time of the experiments. On the other hand, since the excavation force at the tip of the bucket consists of only horizontal and vertical force components (x and y directions) and no torque about z-axis, with good approximation we can determine τ_3 in terms of τ_1 and τ_2. In reality, however, the third component is not zero, because the excavation forces stem from a number of components, including the friction to the body of the bucket. These components, nevertheless, constitute only a small portion of the total excavation force, the main part of which is due to the resistance of the medium to cutting.

The results shown correspond to five excavation runs. In automatic mode, the excavator can be commanded to keep the bucket velocity constant during excavation. The orientation of the bucket is also kept constant with respect to the ground. In this respect, the trajectory of the bucket motion is more or less the same. In the following figures, only actual motion through the ground is presented.

Figure 5 illustrates a typical example of the forces measured by the pressure sensors in the rams. The relative variation of the measured forces for all the five runs can be seen in figures 6 (boom ram) and 7 (dipper ram) A typical case of the calculated horizontal and vertical components of the excavation force is depicted in figure 8. Finally figures 9 and 10, respectively illustrate the minimum and maximum force components experienced during the five experiments.

In the figures shown, the horizontal axis shows the sampling number of the measurement taken, and is proportional to time. The values on the vertical axis in figures 5, 6 and 7 denote the voltage of the signal from the pressure transducer. A reading of .5 volt represents zero pressure and 4.5 volt corresponds to a pressure of 350 bars. The force values in figures 8 to 10 are in Newton.

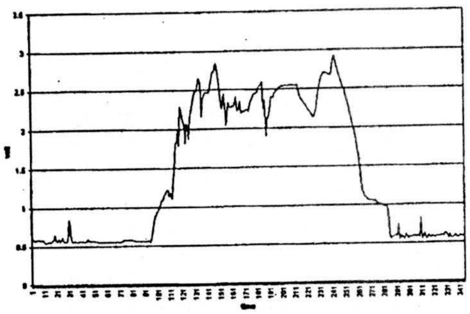

Figure 5- A typical reading of boom ram pressure

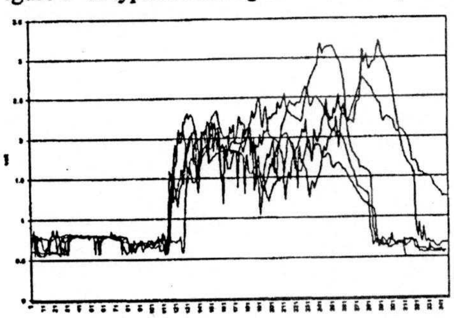

Figure 6- Pressure variation in the boom ram - all runs

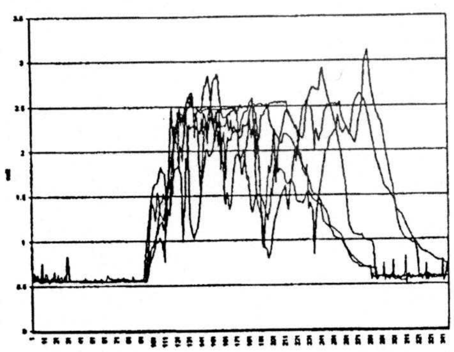

Figure 7- Pressure variation in the dipper ram- all runs

DISCUSSION

The desired outcome of the study is that, for every point on a given trajectory of the cutting edge of the bucket there should be a maximum and minimum value of excavation force, for a given bucket and a specific medium. The results obtained fulfil this requirement to a certain degree, only. The problem lies in the fact that the speed of operation also has an effect on the results. As it can be seen from the figures 9 and 10, the starting points coincide whereas the finishing times are not necessarily the same However, as the number of trials is increased, this difference becomes less and less significant. Consequently, it can be said that the assumption of maximum and minimum values of the excavation force, is correct.

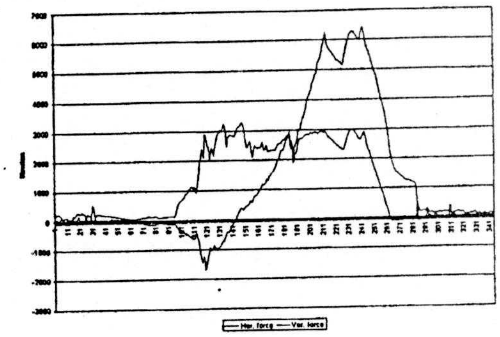

Figure 8- A typical sample of excavation force

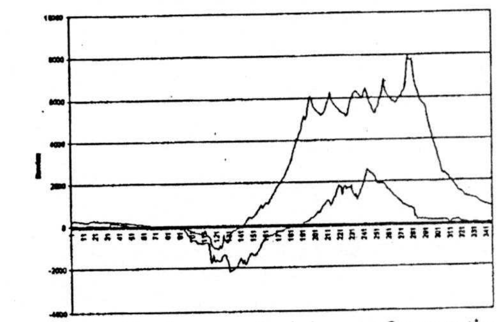

Figure 9 – Maximum and minimum of excavation force horizontal component

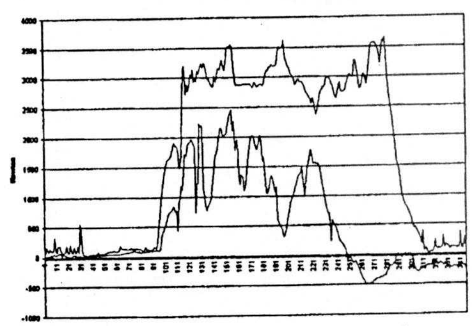

Figure 10 – Maximum and minimum of excavation force vertical component

SUMMARY

This paper presents some analytical results for the kinematics and force relationships in a typical backhoe. These relationships are required for the feed-back purposes at both higher and lower level control of the motion of bucket for automating the excavation process of such a machine. Also, experimental results of the measurement of the excavation force are presented that confirm an assumption of previous work regarding the variation of such a force during an excavation task.

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APPENDIX A

The general solutions to a trigonometric equation of the form

$$m\sin x + n\cos x = h$$

is obtained by, first, writing the equation in the equivalent form

$$\frac{m\sin x}{\sqrt{m^2+n^2}} + \frac{n\cos x}{\sqrt{m^2+n^2}} = \frac{h}{\sqrt{m^2+n^2}}$$

Setting

$$\frac{m}{\sqrt{m^2+n^2}} = \cos y \quad \textit{and} \quad \frac{n}{\sqrt{m^2+n^2}} = \sin y$$

leads to

$$\sin(x+y) = \frac{h}{\sqrt{m^2+n^2}} \quad \textit{and} \quad \cos(x+y) = \frac{\pm\sqrt{m^2+n^2-h^2}}{\sqrt{m^2+n^2}}$$

and the value of the unknown x is, then

$$x = \tan^{-1}\left(\frac{h}{\pm\sqrt{m^2+n^2-h^2}}\right) - y$$

APPENDIX B

In the set of equations (9) and (10) x and y are known and q_1 and q_2 are the two unknowns. can be solved as follows. By rearranging the equations we have

$$x - a_1\cos q_1 = a_2\cos(q_1+q_2)$$
$$y - a_1\sin q_1 = a_2\sin(q_1+q_2)$$

Squaring each side and adding together leads to

$$x^2 + y^2 - 2a_1[y\sin q_1 + x\cos q_1] + a_1^2 = a_2^2$$

or

$$y\sin q_1 + x\cos q_1 = \frac{x^2+y^2+a_1^2-a_2^2}{2a_1}$$

This equation is in the same form as solved in Appendix A. When solved then the magnitude of q_2 can, further, be readily found.